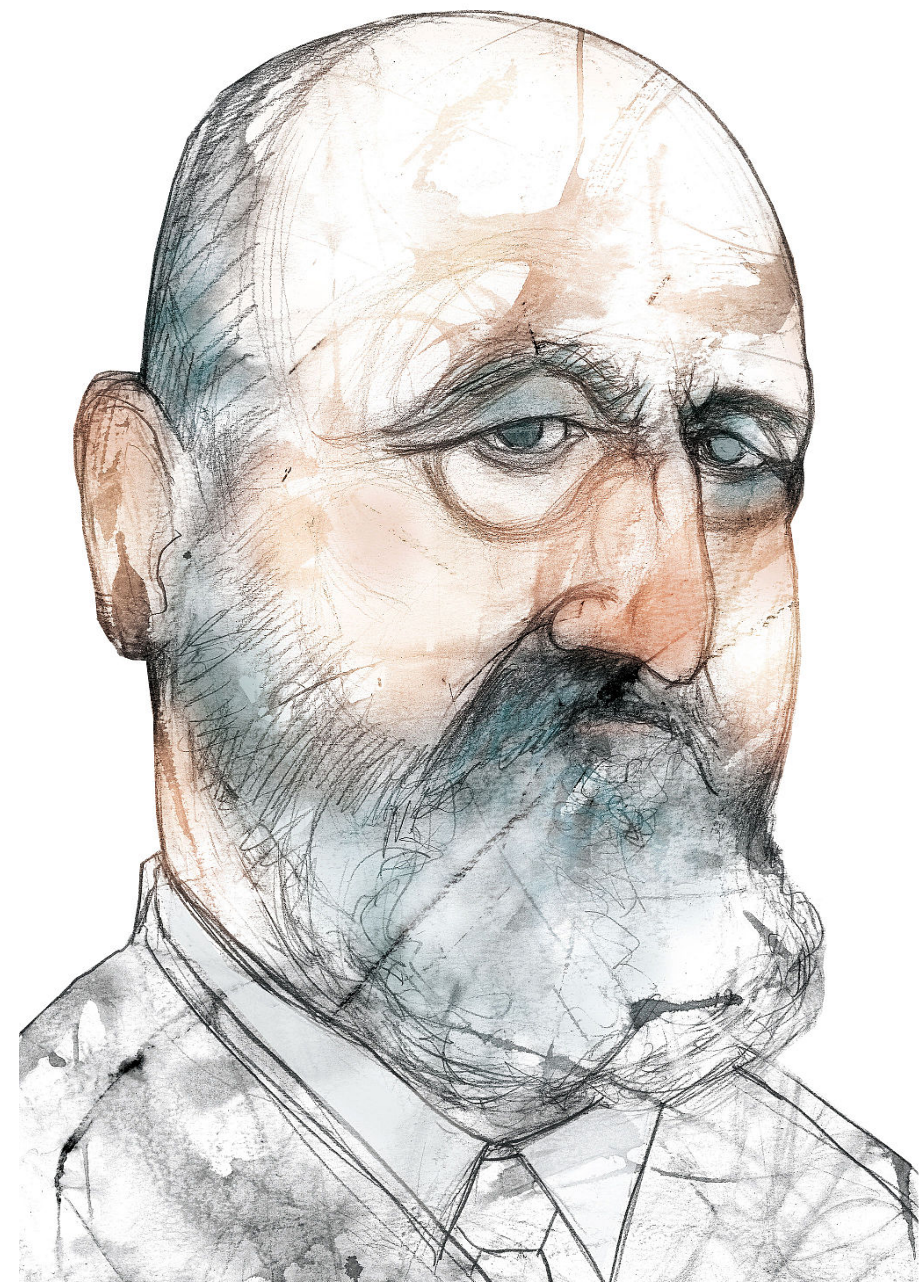
Formatting floating-point numbers

Victor Zverovich

The origins

- Floating point arithmetic was "casually" introduced in 1913 paper "*Essays on Automatics*" by Leonardo Torres y Quevedo, a Spanish civil engineer and mathematician

- Included in his 1914 electro-mechanical version of Charles Babbage's Analytical Engine



Portrait of Torres Quevedo by Eulogia Merle
(Fundación Española para la Ciencia y la Tecnología / CC BY-SA 4.0)

A bit of history

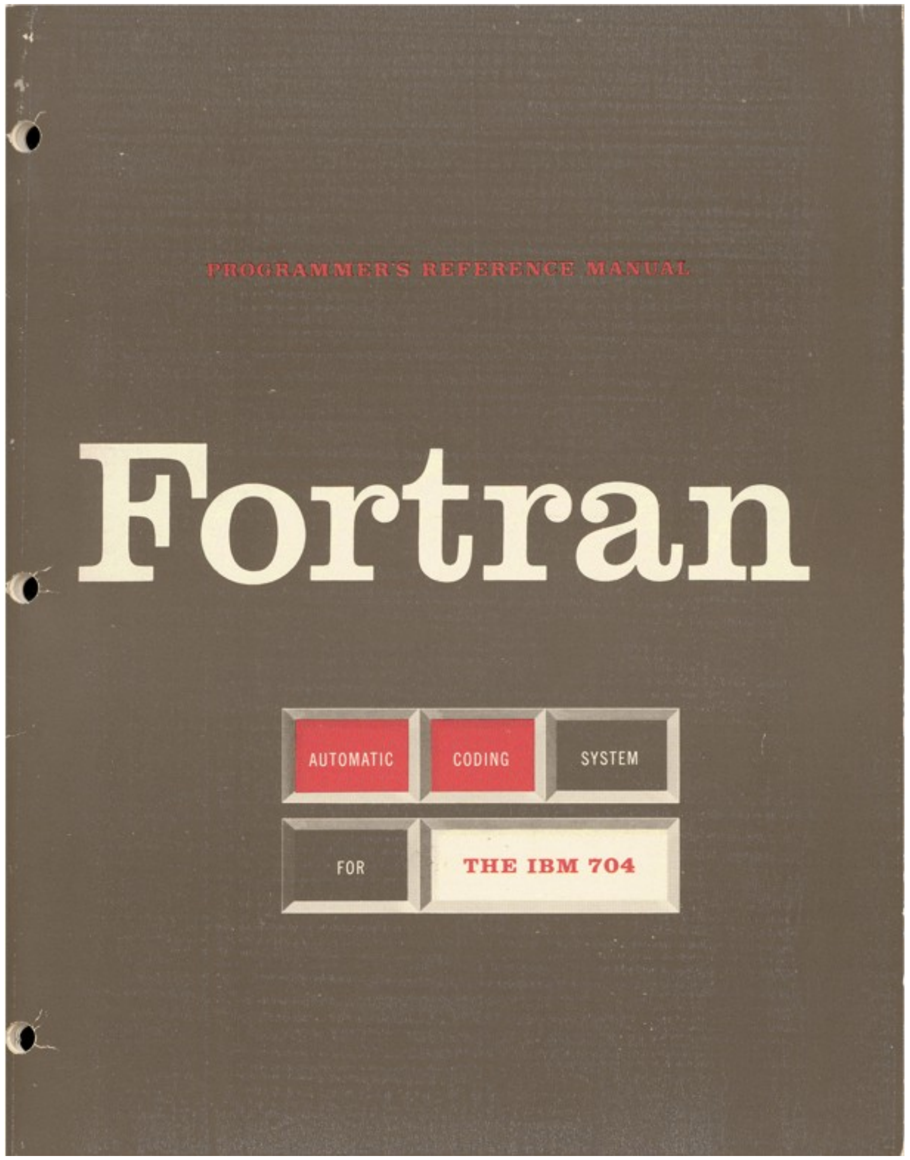
- 1938 Z1 by Konrad Zuse used 24-bit binary floating point
- 1941 relay-based Z3 had +/- infinity and exceptions (sort of)
- 1954 mass-produced IBM 704 introduced biased exponent

Replica of the Z1 in the German Museum of Technology in Berlin

Formatted I/O

FORTRAN had formatted floating-point I/O in 1950s (same time as comments were invented!):

```
      WRITE OUTPUT TAPE 6, 601, IA, IB, IC, AREA
 601  FORMAT (4H A= ,I5,5H  B= ,I5,5H  C= ,I5,
     &        8H  AREA= ,F10.2, 13H SQUARE UNITS)
```



Cover of The Fortran Automatic Coding System for the IBM 704 EDPM (public domain)

FP formatting in C

The C Programming Language, K&R (1978):

```
/* print Fahrenheit-Celsius table
      for f = 0, 20, ..., 300 */
main()
{
      int lower, upper, step;
      float fahr, celsius;

      lower = 0;      /* lower limit of temperature table */
      upper = 300;    /* upper limit */
      step = 20;      /* step size */

      fahr = lower;
      while (fahr <= upper) {
            celsius = (5.0/9.0) * (fahr-32.0);
            printf("%4.0f %6.1f\n", fahr, celsius);
            fahr = fahr + step;
      }
}
```

Still compiles in 2019: https://godbolt.org/z/KsOzjr

Solved problem?

- Floating point has been around for a while
- Programmers have been able to format and output FP numbers since 1950s
- Solved problem
- We all go home now

Solved problem?

- Floating point has been around for a while
- Programmers have been able to format and output FP numbers since 1950s
- Solved problem
- We all go home now
- Not so fast

Solved problem?

References

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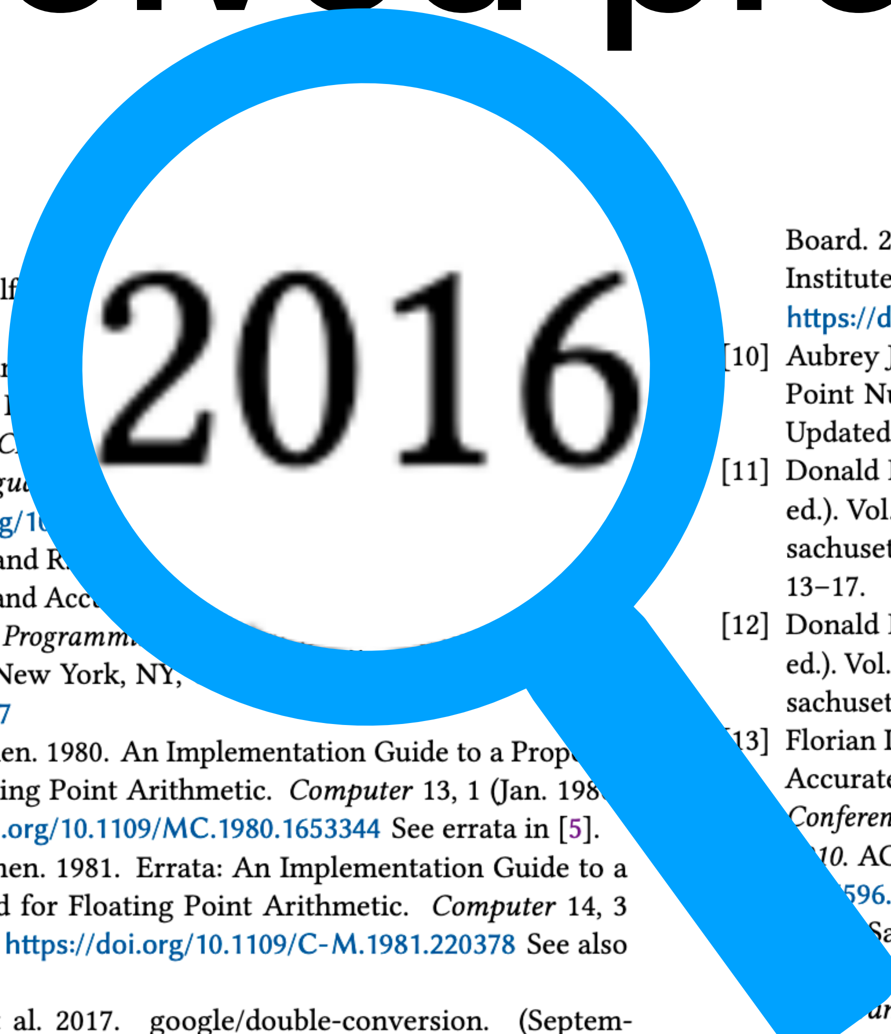

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Meanwhile in 2019

- Neither `stdio`/`printf` nor iostreams can give you the shortest decimal representation with round-trip guarantees
- Performance has much to be desired, esp. with iostreams (can be 3x slower than `printf`!)
- Relying on global locale leads to subtle bugs, e.g. JSON-related errors reported by French but not English users

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:-(

Is floating point math broken?

2538

946

Consider the following code:

```
0.1 + 0.2 == 0.3  ->  false
```

```
0.1 + 0.2         ->  0.30000000000000004
```

Why do these inaccuracies happen?

math language-agnostic floating-point floating-accuracy Edit tags

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Why do these inaccuracies happen?

math language-agnostic floating-point floating-accuracy Edit tags

0.30000000000000004

- Floating-point math is not broken, but can be tricky
- Formatting defaults are broken or at least suboptimal in C & C++ (loose precision):

```
std::cout << (0.1 + 0.2) << " == " << 0.3 << " is "
          << std::boolalpha << (0.1 + 0.2 == 0.3) << "\n";
```

prints "`0.3 == 0.3 is false`"

- The issue is not specific to C++ but some languages have better defaults: https://0.30000000000000004.com/

Desired properties

<u>Steele & White (1990)</u>:

1. No information loss

2. Shortest output

3. Correct rounding

(public domain)

4. ~~Left to right generation~~ - irrelevant with buffering

No information loss

Round trip guarantee: parsing the output gives the original value.

Most libraries/functions lack this property unless you explicitly specify big enough precision: C stdio, C++ iostreams & `to_string`, Python's `str.format` until version 3, etc.

```
double a = 1.0 / 3.0;
char buf[20];
sprintf(buf, "%g", a);
double b = atof(buf);
assert(a == b);

// fails:
// a == 0.3333333333333333
// b == 0.333333
```

```
double a = 1.0 / 3.0;

auto s = fmt::format("{}", a);
double b = atof(s.c_str());
assert(a == b);

// succeeds:
// a == 0.3333333333333333
// b == 0.3333333333333333
```

Shortest output

The number of digits in the output is as small as possible.

It is easy to satisfy the round-trip property by printing unnecessary "garbage" digits (provided correct rounding):

```
sprintf("%.17g", 0.1);
```

prints "`0.10000000000000001`"

```
fmt::print("{}", 0.1);
```

prints "`0.1`"

Correct rounding

- The output is as close to the input as possible.
- Most implementations have this, but MSVC/CRT is buggy as of 2015 (!) and possibly later (both from and to decimal):
  - https://www.exploringbinary.com/incorrect-round-trip-conversions-in-visual-c-plus-plus/
  - https://www.exploringbinary.com/incorrectly-rounded-conversions-in-visual-c-plus-plus/
  - Had to disable some floating-point tests on MSVC due to broken rounding in `printf` and iostreams

<charconv>

- C++17 introduced `<charconv>`
- Low-level formatting and parsing primitives: `std::to_chars` and `std::from_chars`
- Provides shortest decimal representation with round-trip guarantees and correct rounding!
- Locale-independent!

C++ users after <charconv> has been voted into C++17

(public domain)

to_chars

```
std::array<char, 20> buf; // What size?
std::to_chars_result result =
  std::to_chars(buf.data(), buf.data() + buf.size(), M_PI);
if (result.ec == std::errc()) {
  std::string_view sv(buf.data(), result.ptr - buf.data());
  // Use sv.
} else {
  // Handle error.
}
```

- `to_chars` is great, but
- API is too low-level
    - Manual buffer management, doesn't say how much to allocate
    - Error handling is cumbersome (slightly better with structured bindings)
- Can't portably rely on it any time soon. May be widely available in 5 years or so (YMMV).

<format> & {fmt}

- ISO C++ standard paper P0645 proposes a high-level formatting facility for C++20 (`std::format` and friends)

- Implemented in the {fmt} library: https://github.com/fmtlib/fmt

- The default is the shortest decimal representation with round-trip guarantees and correct rounding (will be enabled in the next release)

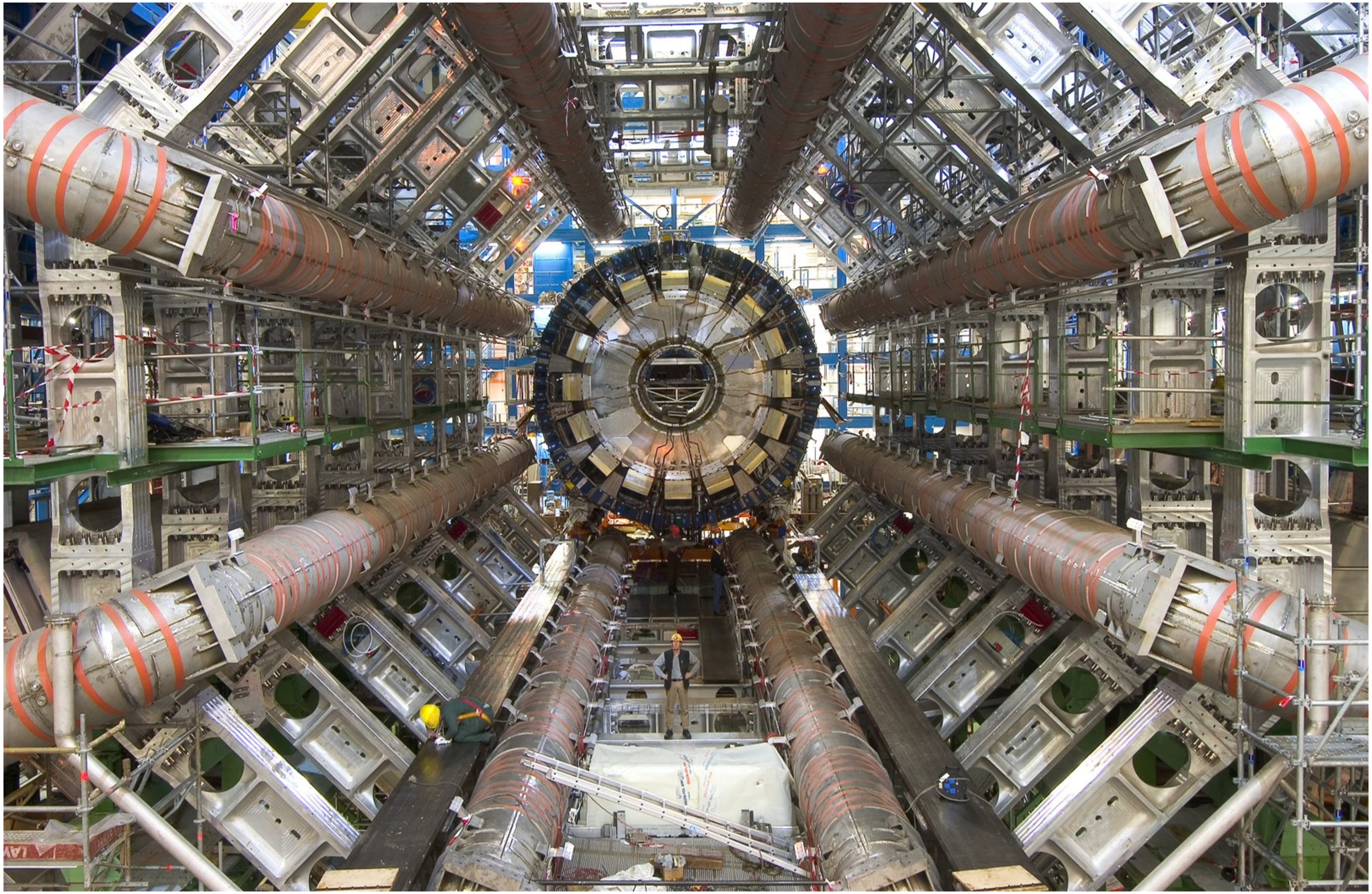
- Control over locales: locale-independent by default

- Example:

```
fmt::print("{} == {} is {}\n", 0.1 + 0.2, 0.3, 0.1 + 0.2 == 0.3);
```

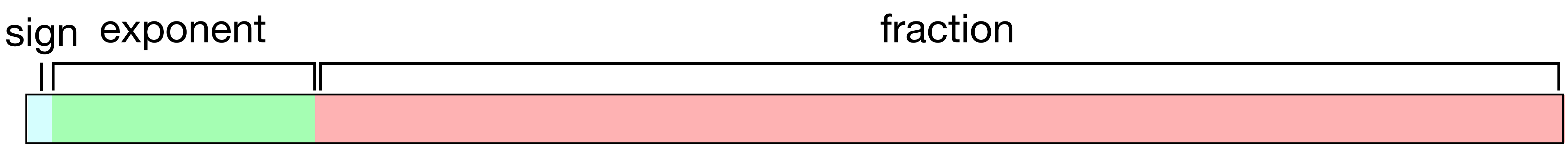
prints `"0.30000000000000004 == 0.3 is false"` (no data loss)

How does it work?



IEEE 754

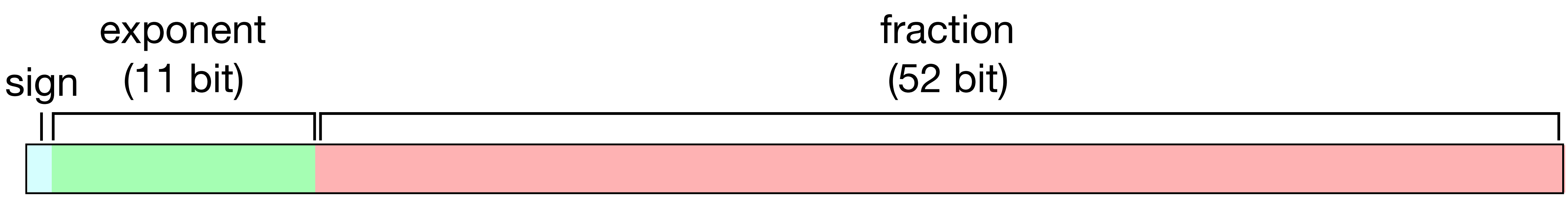
Binary floating point bit layout:



$$
v = \begin{cases} (-1)^{\text{sign}} 1.\text{fraction} \times 2^{(\text{exponent}-\text{bias})} & \textbf{if } 0 < \text{exponent} < 1...1_2 \\ (-1)^{\text{sign}} 0.\text{fraction} \times 2^{(1-\text{bias})} & \textbf{if } \text{exponent} = 0 \\ (-1)^{\text{sign}} \text{Infinity} & \textbf{if } \text{exponent} = 1...1_2, \text{fraction} = 0 \\ \text{NaN} & \textbf{if } \text{exponent} = 1...1_2, \text{fraction} \neq 0 \end{cases}
$$

IEEE 754

Double-precision binary floating point bit layout:



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where `bias = 1023`

Example

π approximation as `double` (`M_PI`):

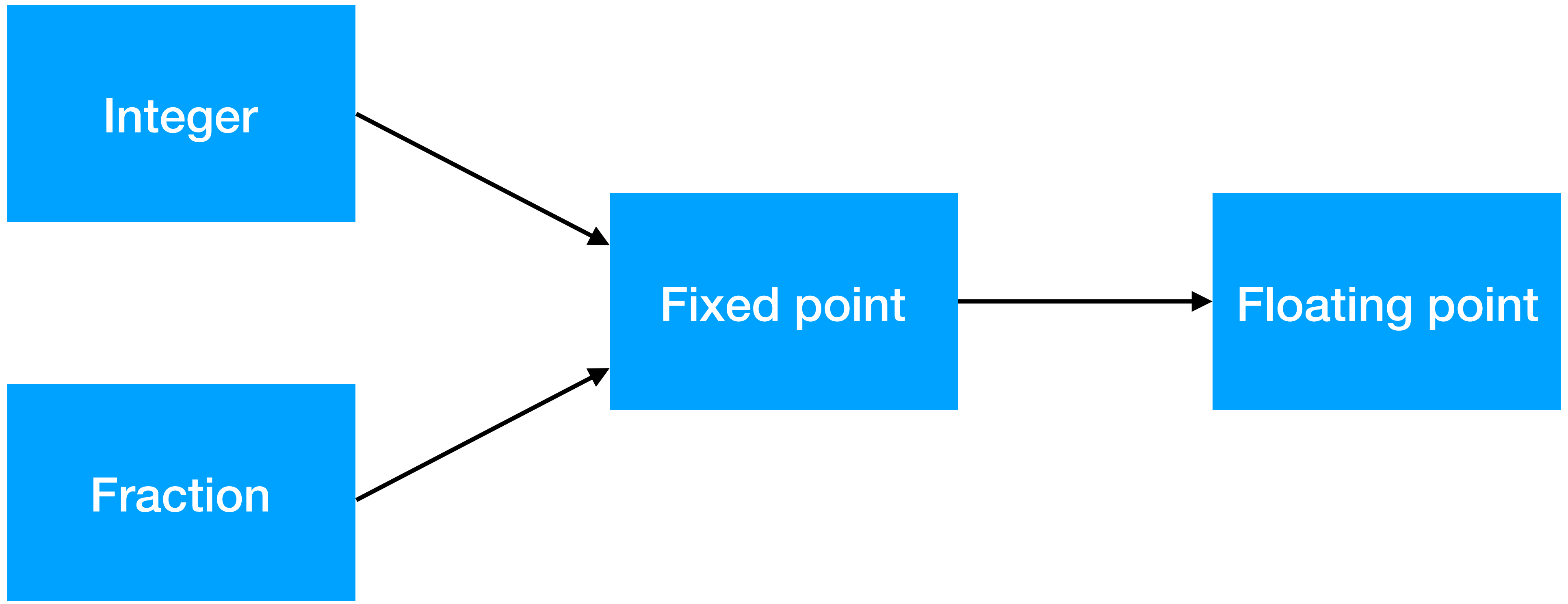
sign	exponent (11 bit)	fraction (52 bit)
0	10000000000	1001001000011111101101010100010001000010110100011000

$$v = (-1)^0 1.1001001000011111101101010100010001000010110100011000_2 \times 2^{(10000000000_2 - 1023_{10})} =$$

$$1.1001001000011111101101010100010001000010110100011_2 \times 2 =$$

$$11.001001000011111101101010100010001000010110100011_2$$

Building up



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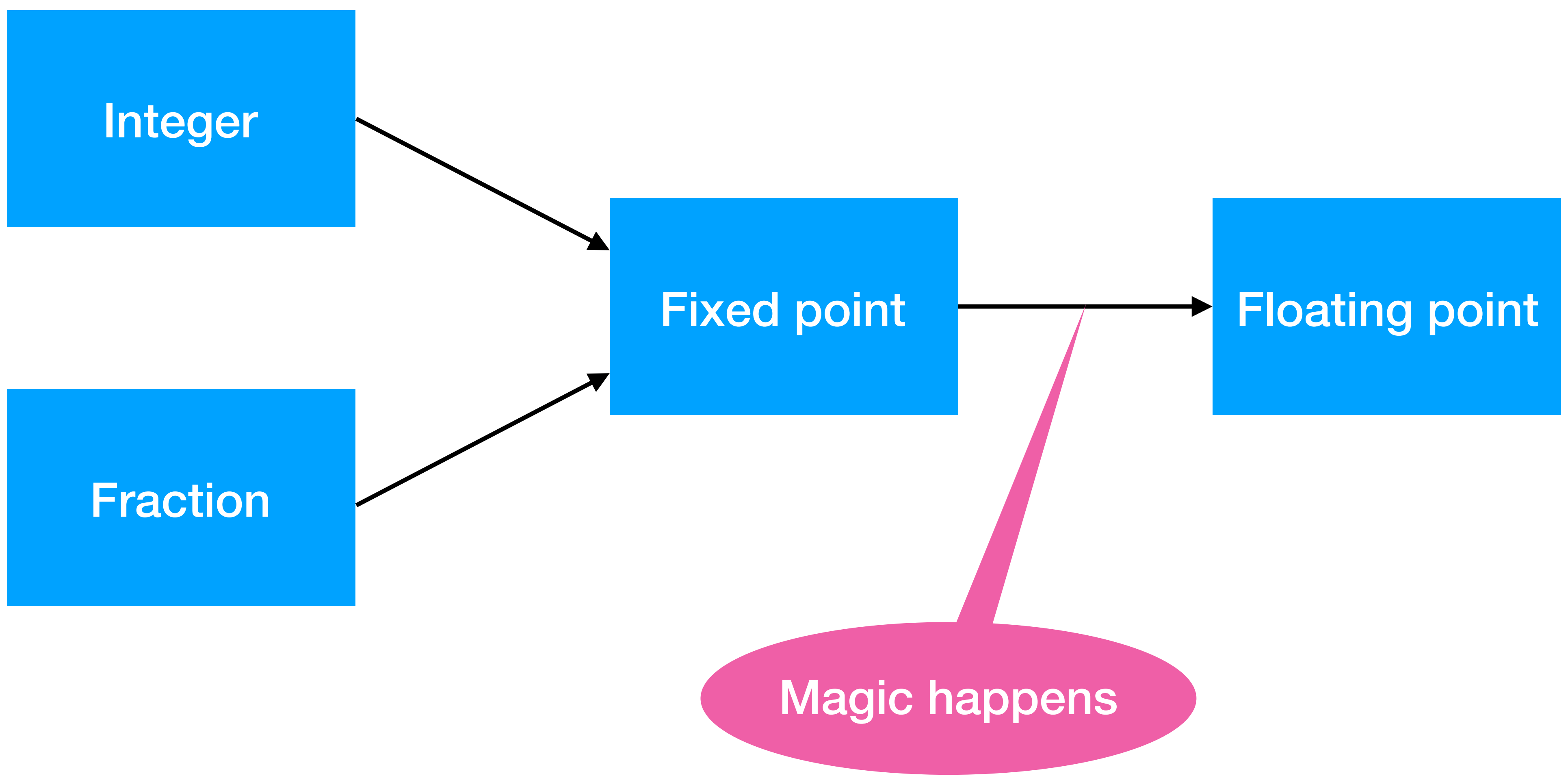
Integer
Fraction
Fixed point
Floating point
Magic happens

Integers

```
constexpr uint64_t uint64_max_digits =
  std::numeric_limits<uint64_t>::digits10 + 1;

char* format_integer(char* out, uint64_t n) {
  char buf[uint64_max_digits];
  char *p = buf;
  do {
    *p++ = '0' + n % 10;
    n /= 10;
  } while (n != 0);
  do {
    *out++ = *--p;
  } while (p != buf);
  return out;
}
```

Fractions

```
constexpr int max_precision = 17;

// Format a fraction (without "0.") stored in num_bits lower
// bits of n.
char* format_fraction(char* out, uint64_t n, int num_bits,
                      int precision = max_precision) {
 auto mask = (uint64_t(1) << num_bits) - 1;
 for (int i = 0; i < precision; ++i) {
    n *= 10;
    *out++ = '0' + (n >> num_bits); // n / pow(2, num_bits)
    n &= mask;                      // n % pow(2, num_bits)
  }
  return out;
}
```

Why 17?

- "17 digits ought to be enough for anyone"
  — some famous person

- *In-and-out conversions*,
  David W. Matula (1968):

  Conversions from base *B* round-trip through base *v* when $B^n < v^{m-1}$, where n is the number of base *B* digits, and m is the number of base *v* digits.

$$\lceil \log_{10}(2^{53}) + 1 \rceil = 17$$

Photo of a random famous person
(public domain)

Fractions

fractional part of `M_PI` (51 bits)

`000000000000.001001000011111101101010100010001000010110100011000`

```
char buf[max_precision + 1];
format_fraction(
  buf,
  0b001'0010'0001'1111'1011'0101'0100'0100'0100'0010'1101'0001'1000,
  51);
```

`buf` contains `"141592653589793111"` - fractional part of `M_PI` (last digit is a bit off, but round trips correctly)

Small exponent

```
// Formats a number represented as v * pow(2, e).
char* format_small_exp(char* out, uint64_t v, int e) {
  auto p = format_integer(out, v >> -e);
  auto int_digits = p - out;
  *p++ = '.';
  auto fraction_mask = (uint64_t(1) << -e) - 1;
  return format_fraction(p, v & fraction_mask, -e,
                         max_precision - int_digits);
}

auto bits = std::bit_cast<uint64_t>(M_PI);
auto fraction_bits = 52, bias = 1023;
auto implicit_bit = uint64_t(1) << fraction_bits;
auto v = (bits & (implicit_bit - 1)) | implicit_bit;
auto e = ((bits >> fraction_bits) & 0x7ff) - bias - fraction_bits;
char buf[max_precision + 1];
format_small_exp(buf, v, e);
```

buf contains `"3.1415926535897931"`


(public domain)

Here be dragons: full exponent, rounding, errors

Exponent

- Full exponent range: 10^{-324} - 10^{308}
- In general requires multiple precision arithmetic
- glibc pulls in a GNU multiple precision library for `printf`:

Overhead	Command	Shared Object	Symbol
57.96%	a.out	libc-2.17.so	[.] __printf_fp
15.28%	a.out	libc-2.17.so	[.] __mpn_mul_1
15.19%	a.out	libc-2.17.so	[.] __mpn_divrem
5.79%	a.out	libc-2.17.so	[.] hack_digit.13638
5.79%	a.out	libc-2.17.so	[.] vfprintf

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Grisù

- Family of algorithms from paper "*Printing Floating-Point Numbers Quickly and Accurately with Integers*" by Florian Loitsch (2004)
- DIY floating point: emulates floating point with extra precision (e.g. 64-bit for double giving 11 extra bits) using simple fixed-precision integer operations
- Precomputes powers of 10 and stores as DIY FP numbers
- Finds a power of 10 and multiplies the number by it to bring the exponent in the desired range
- With 11 extra bits Grisu3 produces shortest result in 99.5% of cases and tracks the uncertain region where it cannot guarantee shortness
- Relatively simple: Grisu2 can be implemented in 300 - 400 LOC incl. optimizations

DIY Floating Point

- DIY floating point:

```
struct fp {
  uint64_t f; // fraction (with explicit 1)
  int e;      // exponent

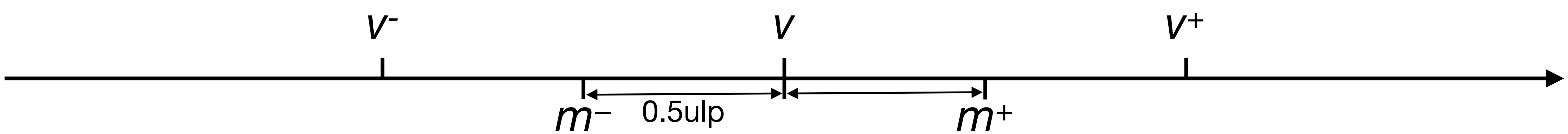
  fp(double d) {
    auto bits = std::bit_cast<uint64_t>(d);
    auto fraction_bits = 52, bias = 1023;
    auto implicit_bit = uint64_t(1) << fraction_bits;
    f = (bits & (implicit_bit - 1)) | implicit_bit;
    e = ((bits >> fraction_bits) & 0x7ff) - bias - fraction_bits;
    // Similarly for denormals
  }
};
```

- $x \otimes y$ - rounded multiplication of DIY FP numbers
- Unit in the last place (ulp) - value of the least significant digit if it is 1.

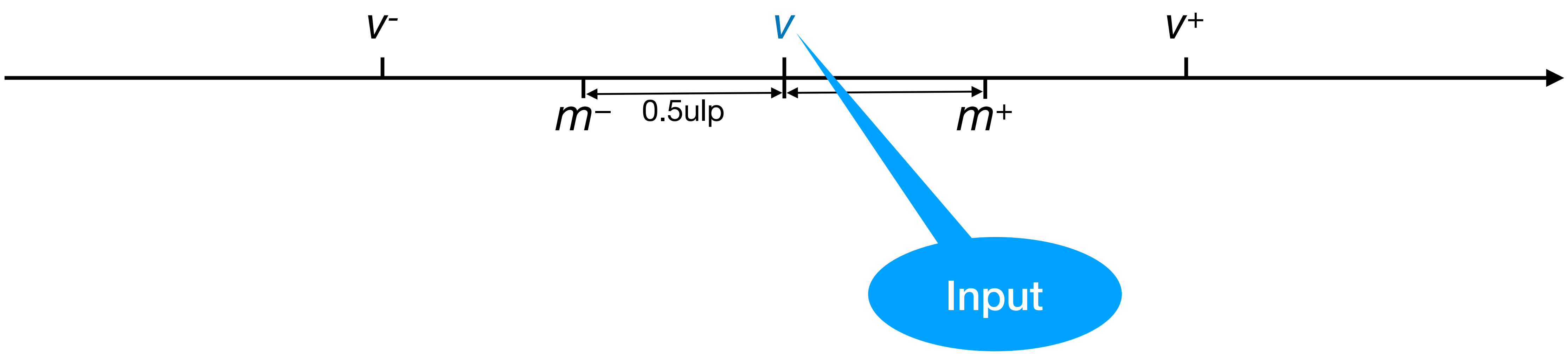
Grisù3

1. ***Boundaries***: given FP number v, compute v's boundaries m^- and m^+.
2. ***Conversion***: convert v, m^-, and m^+ into DIY FPs w, w^-, and w^+ where w and w^+ are normalized, w^- and w^+ have the same exponent.
3. ***Cached Power***: find the normalized power of 10 c_{-k} such that $\alpha \leq c_{-k}.e + w^+.e + q \leq \gamma$, where $[\alpha, \gamma]$ is a desired exponent range such as [-60, -32].
4. ***Product***: compute $M^- := w^- \otimes c_{-k}$ - 1ulp, $M^+ := w^+ \otimes c_{-k}$ + 1ulp, $\Delta := M^+ - M^-$.
5. ***Digit Length***: find the greatest κ such that $M^+ \bmod 10^\kappa < \Delta$
6. ***Round***: compute $W := w \otimes c_{-k}$, and let $W^- := W -$ `1ulp`, and $W^+ := W +$ `1ulp`. Set $P_i := \lfloor M^+ / 10^\kappa \rfloor - i$ for $i \geq 0$. Let m be the greatest integer that verifies $P_m \times 10^\kappa > M^-$. Let u, $0 \leq u \leq m$ be the smallest integer such that $| P_u \times 10^\kappa - W^+ |$ is minimal. Similarly let d, $0 \leq d \leq m$ be the largest integer such that $| P_d \times 10^\kappa - W^- |$ is minimal. If $u \neq d$ return `failure`, else set $P := P_u$.
7. ***Weed***: if not $w^- \otimes c_{-k}$ + 1ulp $\leq P \times 10^\kappa \leq w^+ \otimes c_{-k}$ - 1ulp, return `failure`.
8. ***Output***: define $V := P \times 10^{k+\kappa}$. The decimal digits d_i and n are obtained by producing the decimal representation of P (an integer). Set $K := k + \kappa$, and return it with the n digits d_i.

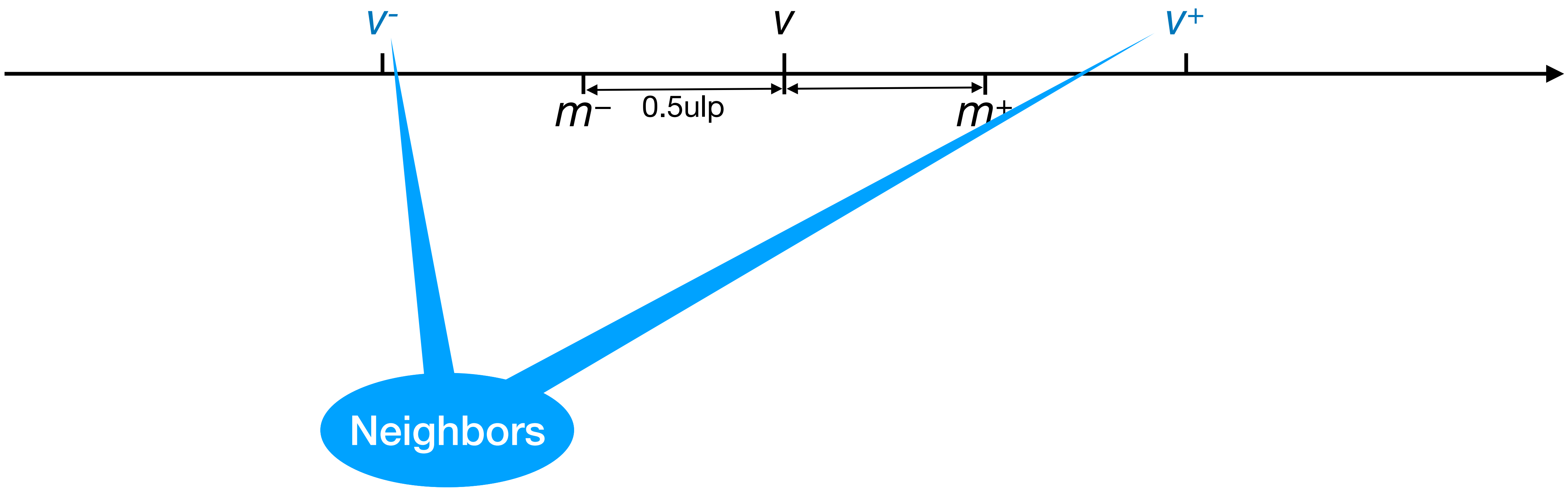
Grisù



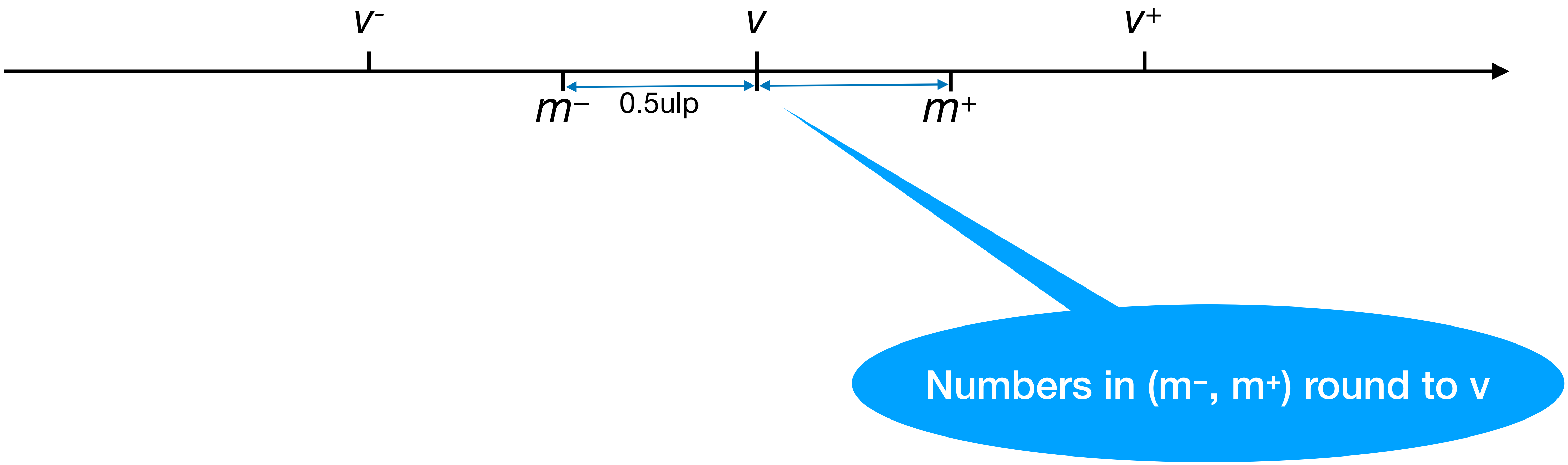
Grisù



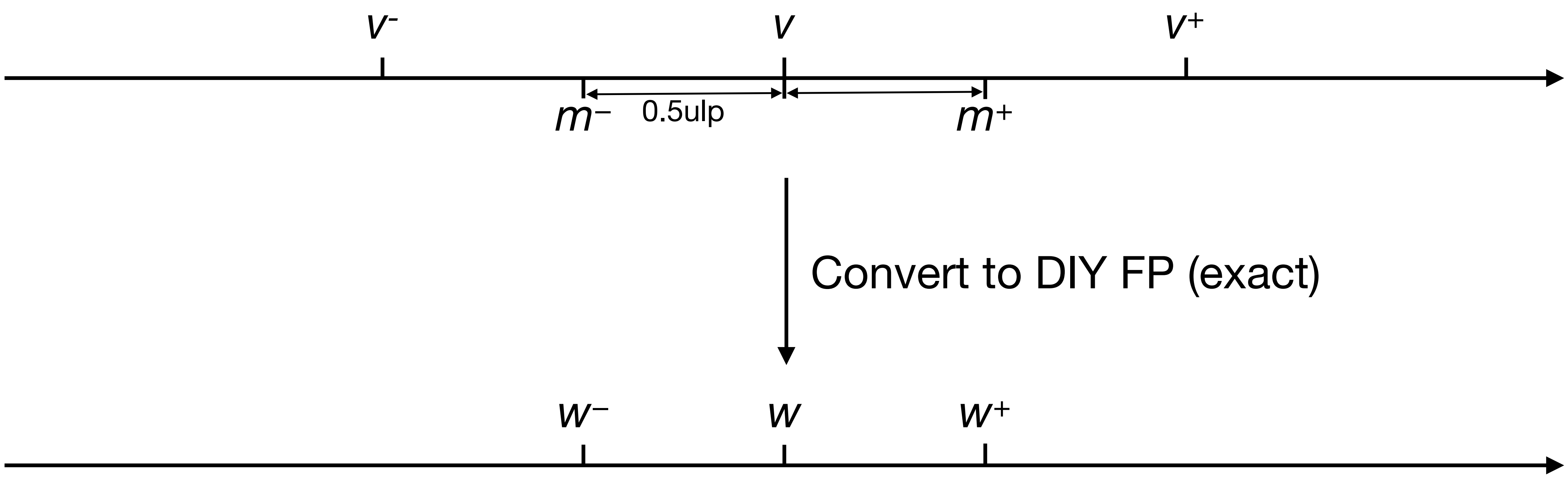
Grisù



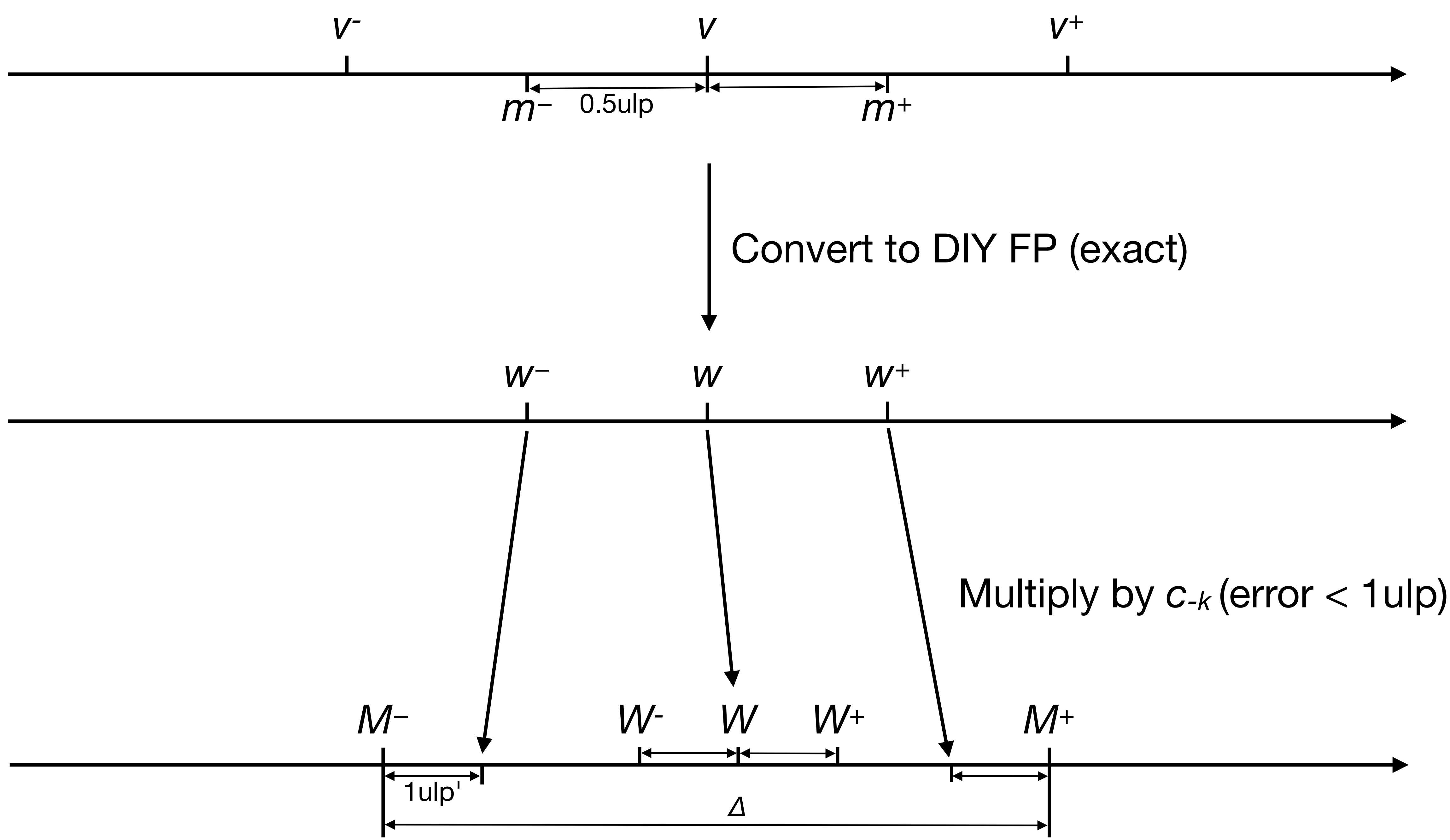
Grisù



Grisù

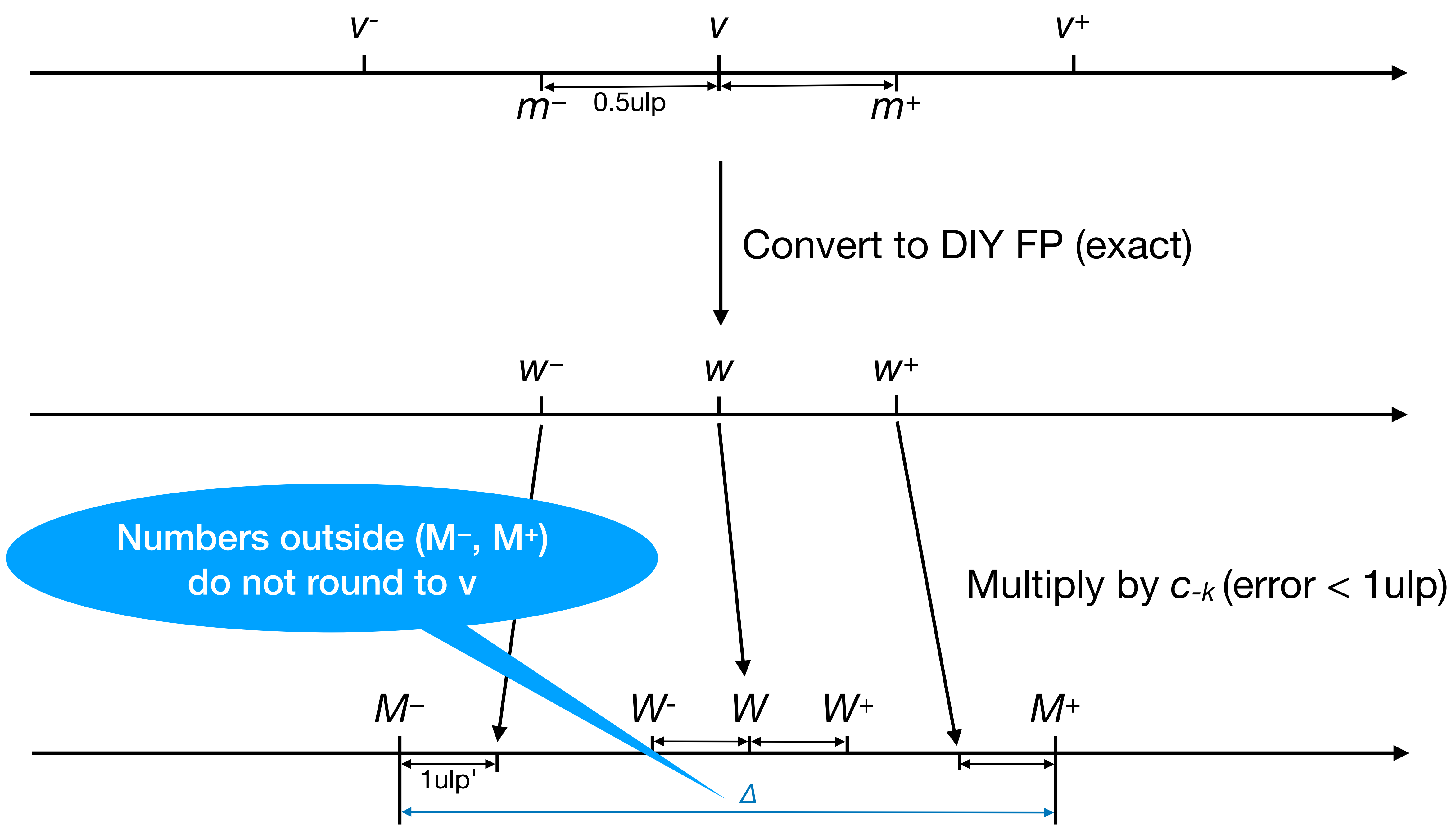


Grisù



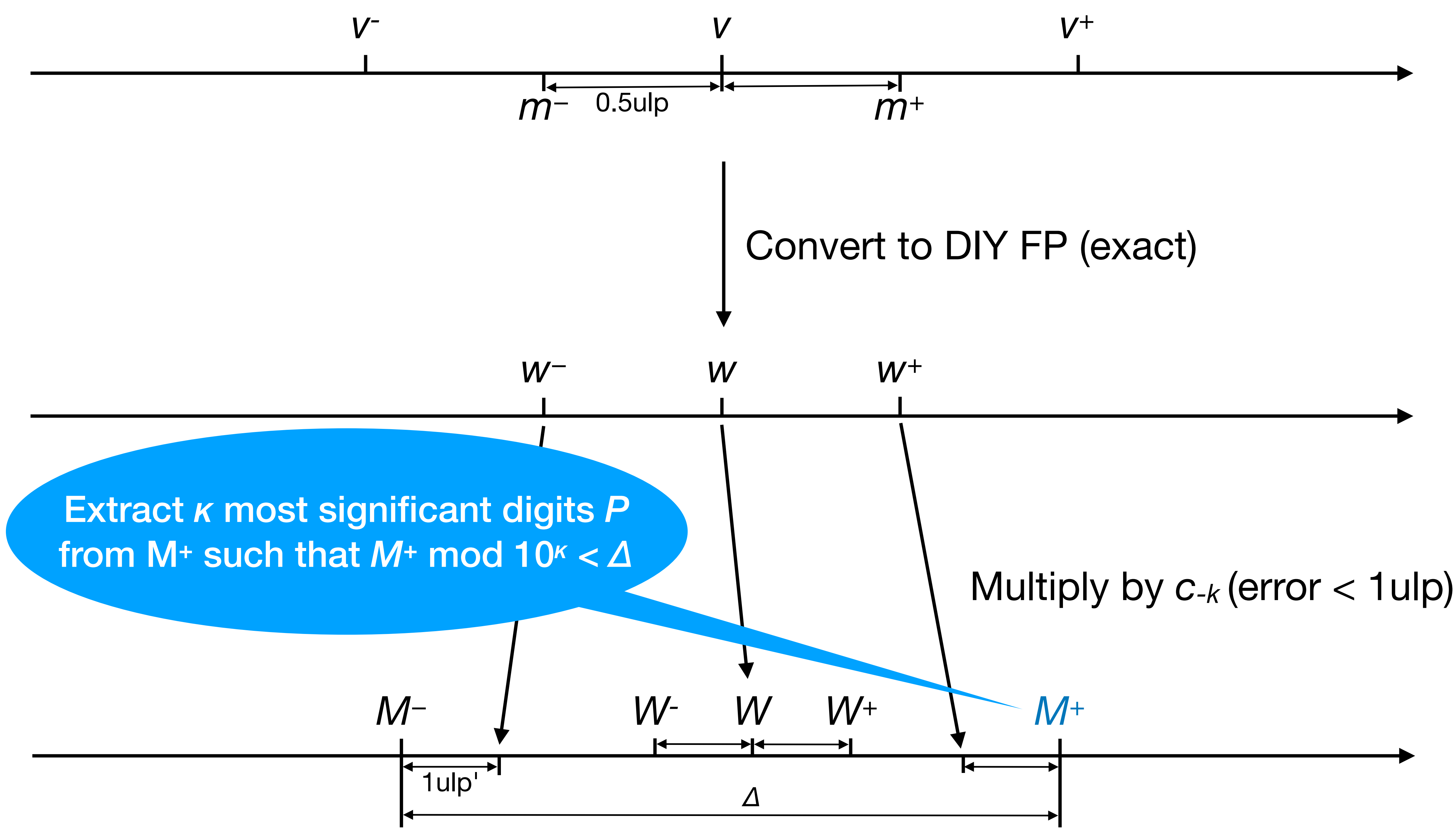
v^-
v
v^+
m^-
0.5ulp
m^+
Convert to DIY FP (exact)
w^-
w
w^+
Multiply by c_{-k} (error < 1ulp)
M^-
W^-
W
W^+
M^+
1ulp'
Δ

Grisù



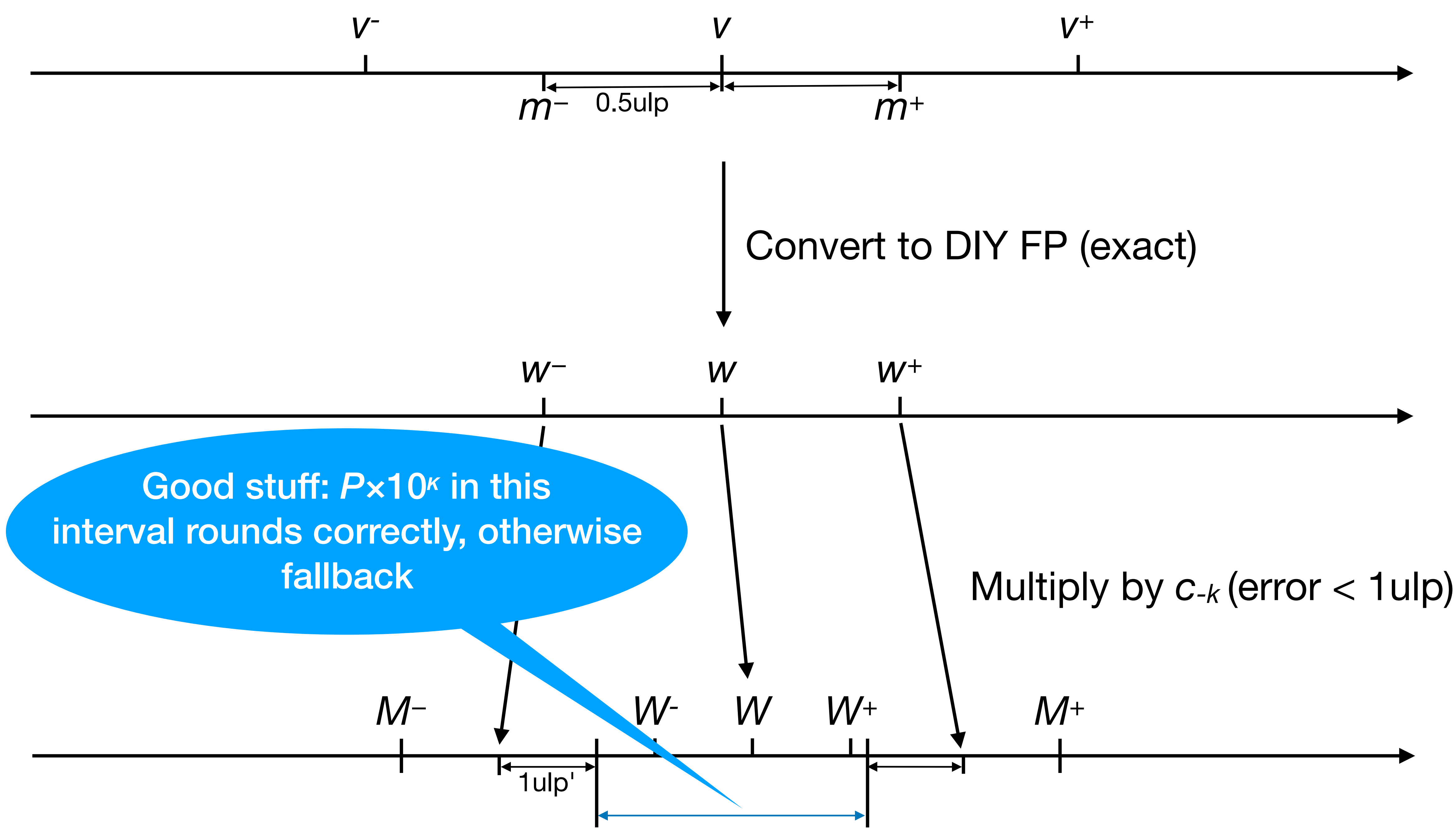
v-
v
v+
m−
0.5ulp
m+
Convert to DIY FP (exact)
w−
w
w+
Numbers outside (M−, M+)
do not round to v
Multiply by c-k (error < 1ulp)
M−
W-
W
W+
M+
1ulp'
Δ

Grisù



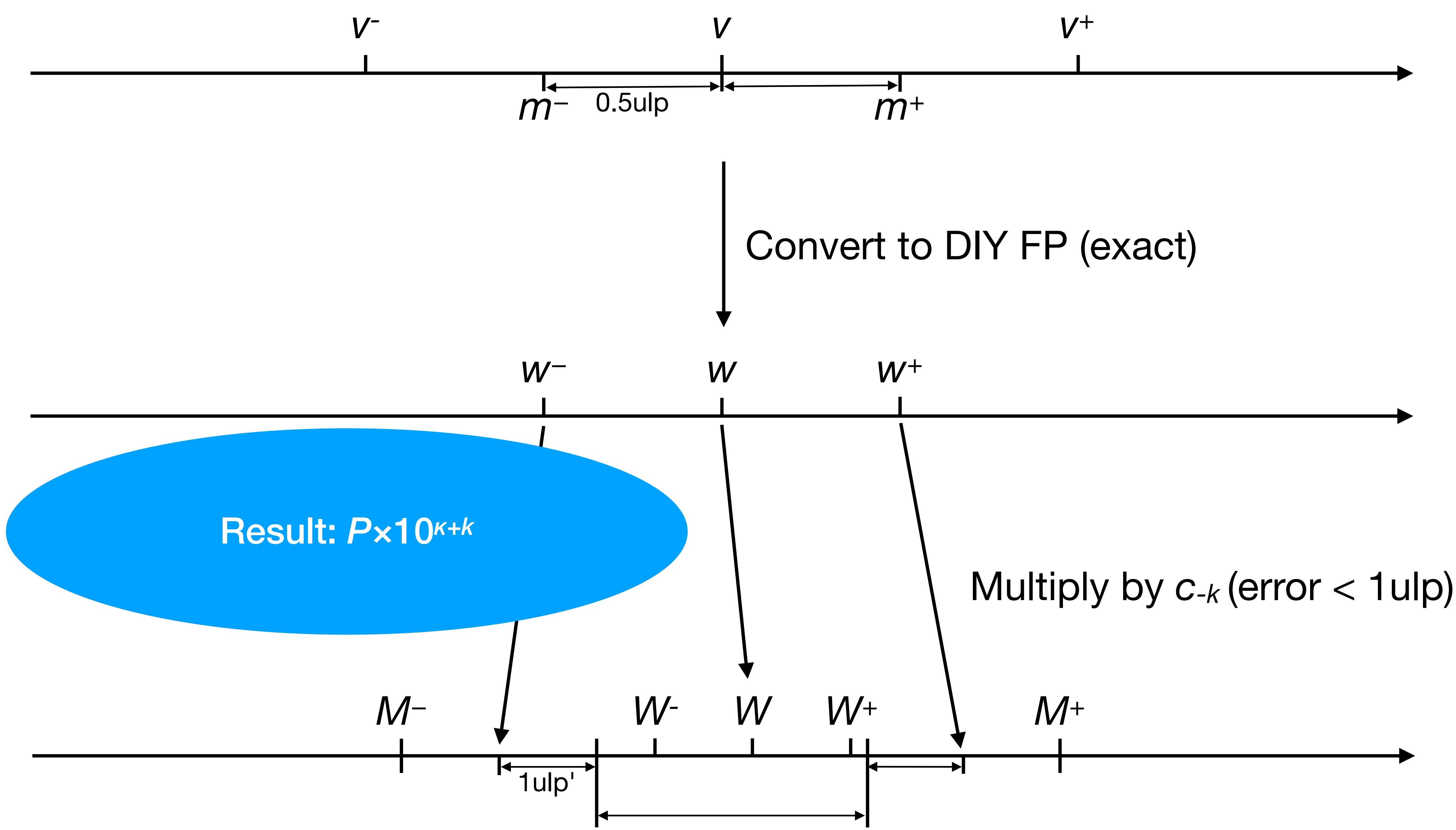
v-
v
v+
m−
0.5ulp
m+
Convert to DIY FP (exact)
w−
w
w+
Extract κ most significant digits P
from M+ such that M+ mod 10κ < Δ
Multiply by c-k (error < 1ulp)
M−
W-
W
W+
M+
1ulp'
Δ

Grisù



v^-
v
v^+
m^-
0.5ulp
m^+
Convert to DIY FP (exact)
w^-
w
w^+
Good stuff: $P \times 10^{\kappa}$ in this interval rounds correctly, otherwise fallback
Multiply by c_{-k} (error < 1ulp)
M^-
W^-
W
W^+
M^+
1ulp'

Grisù



v^-
v
v^+
m^-
0.5ulp
m^+
Convert to DIY FP (exact)
w^-
w
w^+
Result: $P \times 10^{\kappa+k}$
Multiply by c_{-k} (error < 1ulp)
M^-
W^-
W
W^+
M^+
1ulp'

Powers of 10

Generating powers of 10 using Python's built-in arbitrary precision arithmetic (can be easily extended to negative powers scaling by 2^N for some big `N`):

```
min_k = 4
max_k = 340
step = 8
for k in range(min_k, max_k + 1, step):
  binary = '{:b}'.format(10 ** k)
  f = (int('{:0<{}}'.format(binary[:65], 65), 2) + 1) / 2
  e = len(binary) - 64
  print('fp(0x{:016x}, {})'.format(f, e))
```

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```

Output:

```
fp(0x9c40000000000000, -50)
fp(0xe8d4a51000000000, -24)
fp(0xad78ebc5ac620000, 3)
fp(0x813f3978f8940984, 30)
...
```

Optimizations

- Many integer formatting optimization apply, e.g. reducing the number of integer divisions by computing the number of digits with `__builtin_clz` or faster method when processing integral part of M^+

- Use `__builtin_clz` for counting the number of leading zeros when normalizing subnormals

- Use 128-bit integers (e.g. `__uint128_t`) for multiplication by a cached power of 10

- Credit: Milo Yip (GitHub @miloyip)

Ryū

1. Decode the floating point number

2. Compute the interval of information-preserving outputs similar to (m^-, m^+) from Grisu

3. Convert the interval to a decimal power base using precomputed multipliers $\lfloor 2^k / 5^q \rfloor + 1$ and $\lfloor 5^{-(e-2)-q} / 2^k \rfloor$, where e is the exponent

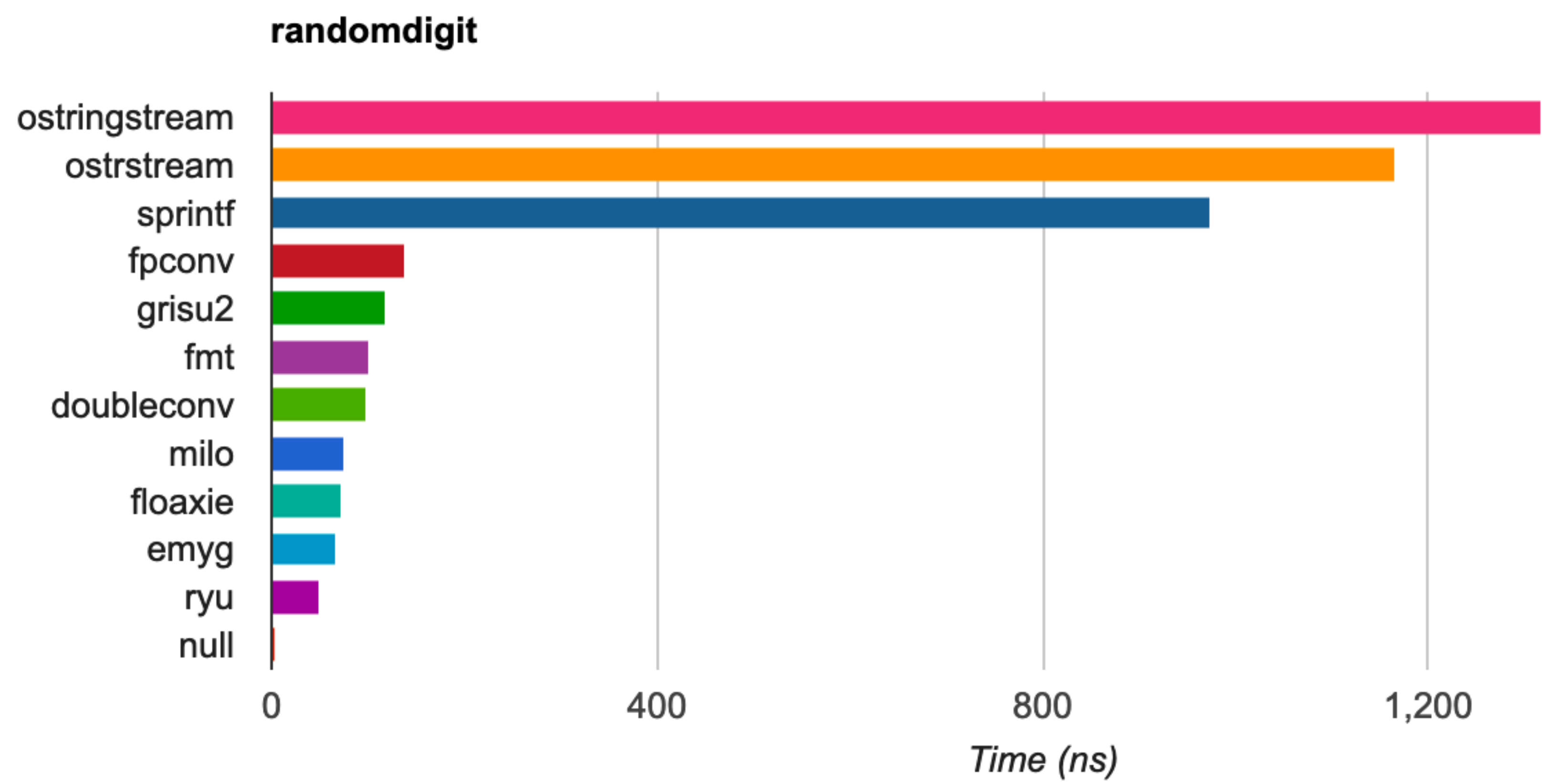
4. Determine the shortest, correctly-rounded string within this interval by repeated division skipping *q* initial iterations by choosing appropriate multipliers in step 3

5. Print

Ryū

- Pros:
  - Doesn't need a fallback to guarantee shortness
  - Faster (~30% compared to a somewhat optimized implementation) than Grisù on random numbers; about the same perf on short output (6 digits or less)
- Cons:
  - Fairly complex
  - Larger precomputed tables: 10 KiB vs 870 bytes for `double`. For comparison all of {fmt} can fit in 30K on some platforms
  - Requires large integer arithmetic e.g. 128-bit for `double` (can be emulated)

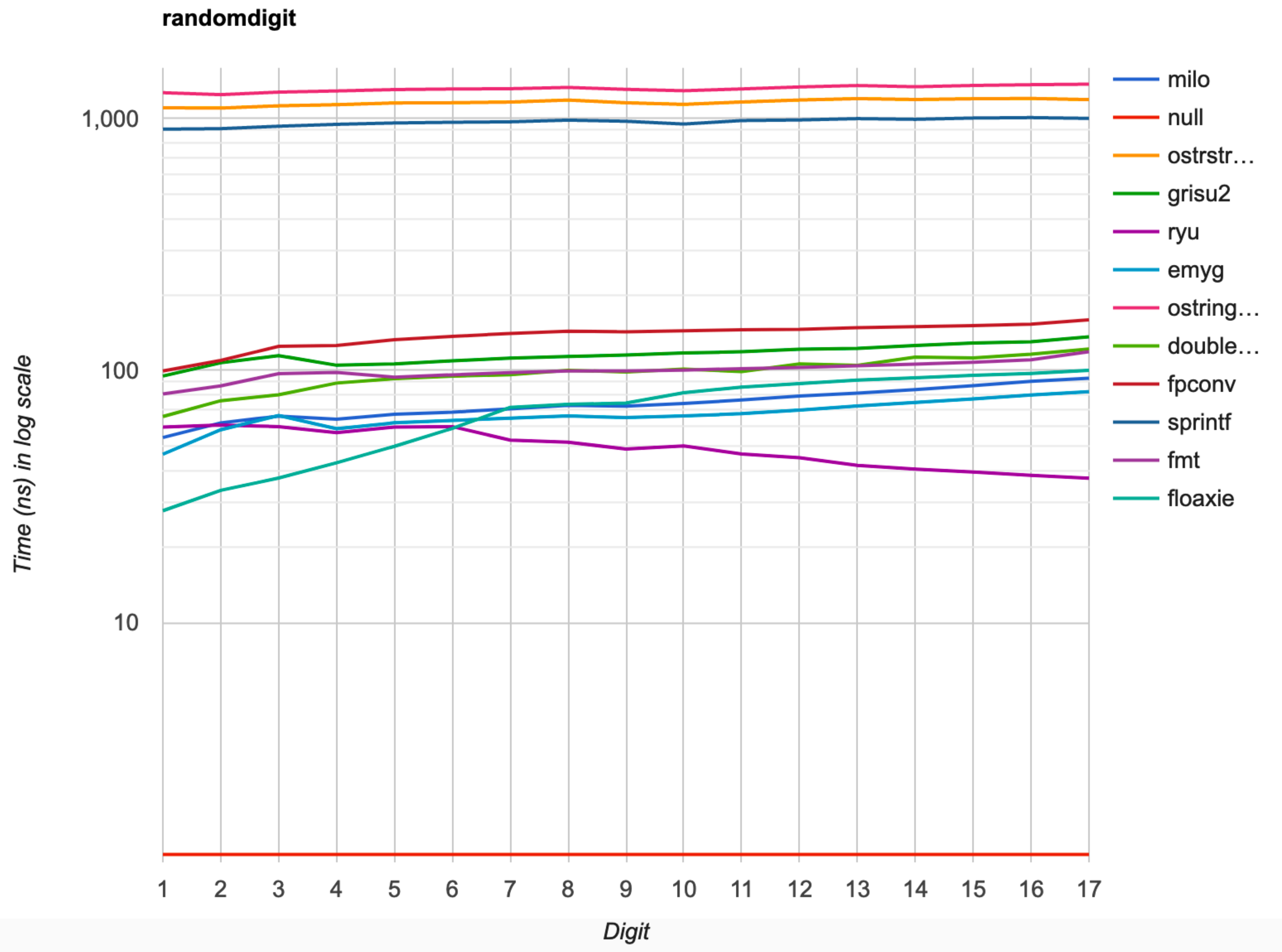
Benchmarks



Function	Time (ns)	Speedup
ostringstrea	1,317.294	1.00x
ostrstream	1,163.747	1.13x
sprintf	971.953	1.36x
fpconv	138.394	9.52x
grisu2	116.565	11.30x
fmt	100.329	13.13x
doubleconv	98.300	13.40x
milo	74.400	17.71x
floaxie	70.965	18.56x
emyg	67.188	19.61x
ryu	50.076	26.31x
null	1.200	1,097.75

https://github.com/fmtlib/dtoa-benchmark
(based on miloyip/dtoa-benchmark)

Bechmarks


randomdigit
Time (ns) in log scale
1,000
100
10
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
Digit
milo
null
ostrstr…
grisu2
ryu
emyg
ostring…
double…
fpconv
sprintf
fmt
floaxie

References

- David W. Matula. 1968. *In-and-out conversions*. Communications of the ACM. Volume 11 Issue 1, Jan. 1968, 47-50.

- Florian Loitsch. 2010. *Printing Floating-Point Numbers Quickly and Accurately with Integers*. In Proceedings of the ACM SIGPLAN 2010 Conference on Programming Language Design and Implementation, PLDI 2010. ACM, New York, NY, USA, 233-243.

- Guy L. Steele Jr. and Jon L. White. 1990. *How to Print Floating-Point Numbers Accurately*. In Proceedings of the ACM SIGPLAN 1990 Conference on Programming Language Design and Implementation (PLDI ’90). ACM, New York, NY, USA, 112-126.

- Ulf Adams. 2018. *Ryū: fast float-to-string conversion*. In Proceedings of the 39th ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI 2018. ACM, New York, NY, USA, 270-282.

- Grisù implementation: https://github.com/google/double-conversion

- Ryū implementation: https://github.com/ulfjack/ryu

Questions?